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Some Results On Generalized Antieigenvalue Pairs and Their Associated Antieigenvectors of Certain Class of Linear Two-Parameter Eigenvalue Problems

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Abstract

In the field of operator algebra, the concept of antieigenvalue theory was first introduced by Karl Gustafson with a special emphasis on accretive operators. In a wide range of scientific applications, antieigenvalue naturally occurs. Due to the inclusion of nonlinear Euler equations in the antieigenvalue theory, computing antieigenvalues is a difficult task as compared to that of computing eigenvalues of the operator. In the current paper, we consider linear two-parameter eigenvalue problems ($\mathbb{LTEP}$) and will discuss the abstract algebraic setting of the problem as proposed by Atkinson. We analyze the generalized antieigenvalue pair and their corresponding generalized antieigenvectors for $\mathbb{LTEP}$ using the consequences of the Cauchy Schwarz inequality. Some generalized antieigenvalue bounds will also be derived. Generalized antieigenvalues and their corresponding generalized antieigenvectors will be calculated solving their relevant optimization problem. For numerical computations, three examples will be provided. Real symmetric matrices are used in the first case, while real diagonal matrices are used in the second case and finally arbitrary matrices are considered.

Keywords: $\mathbb{LTEP}$; generalized antieigenvalues; generalized antieigenvectors; linearization; matrix polynomial.

1 Introduction

Let S be an operator defined on the Hilbert Space H equipped with usual inner product $\langle . \rangle$. Then S is called an accretive operator if $Re \langle Sz, z \rangle \geq 0, \ \forall \ z \neq 0$ (Strictly accretive operator if $Re \langle Sz, z \rangle > 0, \ \forall \ z \neq 0$). For such accretive operators, Karl Gustafson [10], brought into light the concept of antieigenvalue, during his study of certain problems in perturbation theory related to semi-group generators. The first antieigenvalue of S is represented by $\mu_1(S)$ and is defined as,

$$\mu_1(S) = \min_{Sz \neq 0} \frac{Re \langle Sz, z \rangle}{\|Sz\| \, \|z\|}, \tag{1}$$

and the corresponding vector z for which infimum of (1) is attained is characterised as antieigenvector of S. The quantity $\mu_1(S)$, also denoted as $cos(S)$ for being the cosine of the angle of the operator S. Geometrically, $\mu_1(S)$ is the cosine (real cosine) of largest (real) angle through which an arbitrary nonzero vector z can be rotated by the action of the operator S. It was first developed by Gustafson [10, 11] and later by Krein [26] in more independent way. Applications of antieigenvalues are found in diverse scientific domains. Antieigenvalue appears not only in continuum mechanics, economics, but also in number theory [15]. The book [14] contains an extensive overview of the applications of antieigenvalue in operator theory, computational method, wavelet theory, quantum mechanics, as well as in finance and optimization. The works reported in [34, 23] provide the applications of antieigenvalue in statistics and, similarly in the papers [12, 41] contains the applications of antieigenvalue in economics. In his paper [38], Seddighin presented antieigenvalue inequalities among trigonometric quantities for multiple operators.

Guo et al. [9] investigated the application of antieigenvalues to spectrum sensing and designed an antieigenvalue-based detector. For accretive normal operator, Gustafson and Seddighin [16] derived antieigenvalues bounds and they also developed the theory on total antieigenvectors in [17]. Moreover, they also analyzed antigenvalues of accretive compact normal operator in [43] that express the first antieigenvalue and the components of the first antieigenvectors, and developed an algorithm for computing higher antieigenvalues. An overview of antieigenvalues for Hermitian positive definite operators have been reported in [27]. Computation aspects of antieigenvalue have been analyzed in [43], but they are particularly for normal operators. Mirman [27] developed a method based on Toeplitz-Hausdorff theorem to estimate antieigenvalue of strictly accretive operator. Paul et al. [32] computed antieigenvalues of bounded linear operators via Centre of Mass. Seddighin [39] independently developed a method of computation for antieigenvalue of a strictly accretive operator based on the properties of the numerical range of an operator. He also approximated several antieigenvalue-type quantities for arbitrary accretive operators in [42].

Estimation of antieigenvalue bound has been found in [11]. It was Gustafon, who introduced the notion of interaction antieigenvalues in [13], and after that Seddighin [40] introduced joint antieigenvalues of pairs of operators that belong to the same closed normal subalgebra. Paul introduced antieigenvalue and antieigenvectors of the generalized eigenvalue problem in his paper [31]. Khattree [22] extended the concept of smallest antieigenvalue of a real symmetric positive definite matrix to the generalized antieigenvalue of order r, and provided a closed form expression for the generalized antieigenvalue and generalized antieigenmatrix. Hossein et al. [20] developed the theory of symmetric anti-eigenvalue and symmetric anti-eigenvector of a bounded linear operator along with their applications in Statistics. Recent work on L^p-antieigenvalue conditions has been reported in [30] for complex-valued Ornstein-Uhlenbeck operators.

Organization of the paper: Section 2 contains few notation and basic results to be used in this article. Section 3 contains an abstract setting of $\mathbb{LTEP}$. In Section 4, a general theory of generalized antieigenvalue pairs and their corresponding generalized antieigenvectors are presented. In

Section 5, computational aspects of generalized antieigenvalue pair are discussed with the help of numerical example. Lastly, in Section 6, a conclusion is presented on the overall works.

2 Notations and Basic Results

The following basic results and notation will be used throughout the paper. The notations $\mathbb{R}$ and $\mathbb{C}$ denotes respectively, the set of real numbers and complex numbers. A^{-1} and A^T represents respectively, the inverse and transpose of the matrix A. The Euclidean norm of A is represented by $\|A\|$ and the standard Kronecker product is denoted by $\otimes$.

Definition 2.1. [6] *Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{p\times q}$ are any two matrices. Then the Kronecker Product $(.\otimes.)$ of the matrices A and B is defined as $A \otimes B = [a_{ij}B] \in \mathbb{C}^{mp\times nq}$, where a_{ij} is the i^{th} row and j^{th} column element of A.*

Definition 2.2. [22] *The celebrated Cauchy-Schwarz inequality states that for any two vectors p and q, the following inequality is true,*

$$|(p^Tq)^T|^2 \leq p^Tp.q^Tq.$$

The equality holds if and only if the vectors p and q are proportional to each other.

Definition 2.3. [2] *Let H be a Hermitian matrix. Then the matrix H is said to be accretive (or strictly accretive) according as H is positive semi-definite (or positive definite).*

Lemma 2.1. *(Lemma 1, [21]) Consider $0 < a_1 < \ldots < a_4$ and let a_{ij} and b_{ij} be the arithmetic and geometric means of a_i and a_j. Then $\frac{a_{14}}{b_{14}} \geq \frac{a_{23}}{b_{23}}$.*

Lemma 2.2. [8] *Let A be any matrix of order $n \times n$, then $\frac{\partial}{\partial x}(x^TAx) = (A + A^T)x$. If A is symmetric, then $\frac{\partial x^TAx}{\partial x} = 2Ax$.*

3 General Theory of $\mathbb{LTEP}$

Consider the $\mathbb{LTEP}$ given below,

$$\begin{aligned} L_1(\lambda,\mu)x_1 &:= (T_1 - \lambda B_{11} - \mu B_{12})x_1 = 0, \\ L_2(\lambda,\mu)x_2 &:= (T_2 - \lambda B_{21} - \mu B_{22})x_2 = 0, \end{aligned} \tag{2}$$

where $\lambda, \mu \in \mathbb{C}$; $x_i \in \mathbb{C}^{n_i}$; and $T_i, B_{ij} \in \mathbb{C}^{n_i\times n_i}$; $i, j = 1, 2$. If for some λ, μ, there exists $0 \neq x_i; i = 1, 2$ such that they satisfy the system (2), then the pair (λ, μ) is termed as eigenvalue and its associated tensor product $x_1 \otimes x_2$ is termed as the eigenvector (right). Similarly, for $i = 1, 2$, a tensor product $v = v_1 \otimes v_2$ is called a left eigenvector if $v_i \neq 0$ and $v_i^*\mathbb{L}_i(\lambda, \mu) = 0$. It is worth-mentioning that eigenvalue problems have received a lot of attention by the researchers in the recent times [29, 28]. The $\mathbb{LTEP}$ arises most often in many practical applications, such as in modeling of stochastic games [5], in dynamic model updating problem [4] and in other applications reported in [46].

The literature on abstract algebraic setting of multiparameter problem ($\mathbb{LTEP}$ is a special case) is available in the works of Atkinson [1], where he established the relationship between multiparameter problem with a system of joint generalized eigenvalue problems ($\mathbb{GEP}$) in tensor product space. Volkmer [44] developed the spectral theory of multiparanmeter system based on the works of [1]. Hochstenbach and Plestenjak [19] analyzed the backward error, condition numbers, and pseudo-spectrum of multiparameter system and given the basis for the second root subspace of simple eigenvalue of the problem. Kosir [24] presented a completeness theorem for nonderogatory eigenvalues of multiparameter systems in finite-dimensional case. Kosir and Plestenjak [25], provided a comprehensive overview on singular $\mathbb{LTEP}$.

For computation of eigenvalue of $\mathbb{LTEP}$, Cock and Moor [5] developed a unifying framework, where they exploit the properties of block shift-invariant subspaces and also used multi-dimensional realization algorithms. Ringh and Jarlebring [35] used a nonlinearization technique to address a $\mathbb{LTEP}$. Ruymbeek et al. [37] presented a subspace method for multiparameter system based on tensor-train representations. Rodriguez et al. [36] developed a Fiber product homotopy method for multiparameter system and also analyzed the sensitivity of the problem. The MultiParEig [33] package available in MATLAB is widely used toolbox for solving $\mathbb{LTEP}$. Numerical method presented in [7] using alternating method is more recent in literature. Two algorithms developed by Vermeersch and Moor in [45], based on block Macaulay matrix to solve rectangular multiparsameter system can also be used to solve $\mathbb{LTEP}$ when the coefficient matrices are rectangular.

The intensive study presented by Hochstenbach et al. [18] can be used a ready reckoner to address rectangular case, where they solved numerically the rectangular multiparameter system by a transformating into a usual Multiparameter problem. They applied these techniques to calculate the optimal least squares autoregressive moving average (ARMA) model and the optimal least squares realization of autonomous linear time-invariant (LTI) dynamical system. The de-facto method for the spectral analysis of $\mathbb{LTEP}$ is by transforming the problem into a certain commuting tuple of operators matrices defined in (3) and (4),

$$\Delta_0 = B_{11} \otimes B_{22} - B_{12} \otimes B_{21}, \tag{3}$$

$$\begin{aligned} \Delta_1 &= T_1 \otimes B_{22} - B_{12} \otimes T_2, \\ \Delta_2 &:= B_{11} \otimes T_2 - T_1 \otimes B_{21}. \end{aligned} \tag{4}$$

The problem is called nonsingular, when Δ_0 defined in (3) is nonsingular, otherwise the problem is called singular. For spectral analysis the $\mathbb{LTEP}$ the nonsingular case is usually considered by the authors. It is well known that, a nonsingular $\mathbb{LTEP}$ defined in (2) can be transformed into a pair of joint $\mathbb{GEP}$ [1] of the form given below,

$$\begin{aligned} \Delta_1 x &= \lambda \Delta_0 x, \\ \Delta_2 x &= \mu \Delta_0 x. \end{aligned} \tag{5}$$

Denote,

$$\Gamma_i = \Delta_0^{-1} \Delta_i, i = 1, 2. \tag{6}$$

For nonsingular $\mathbb{LTEP}$ the matrices Γ_i commute and eigenvalues of (2) and (5) coincides. A $\mathbb{LTEP}$ is referred as Hermitian, if all the matrices present in the system of (2) are Hermitian. Such a Hermitian $\mathbb{LTEP}$ is termed as Right definite if,

$$\begin{vmatrix} x_1^* B_{11} x_1 & x_1^* B_{12} x_1 \\ x_2^* B_{21} x_2 & x_2^* B_{22} x_2 \end{vmatrix} \geq \alpha, \tag{7}$$

for some scalar $\alpha > 0$ and $\|x_i\| = 1 \; \forall \; x_i \in H_i, i = 1, 2$. On the other hand, it is proved in [1] that Right definiteness condition implies that the determinantal operator Δ_0 is positive definite. Set $N = n_1.n_2$. If $\mathbb{LTEP}$ is Right definite, then there exist N number of linearly independent eigenvectors and all the eigenvalues $\lambda, \mu \in \mathbb{R}$. Furthermore, if all the matrices present in the system of (2) of the Right definite problem are real, then real eigenvectors can be selected. Again, the associated left and right eigenvectors are same for real geometrically simple eigenvalue of a Hermitian $\mathbb{LTEP}$. For decomposable tensor $x = x_1 \otimes x_2$, it follows that,

$$\Delta_0 x = (B_{11}x_1 \otimes B_{22}x_2) - (B_{12}x_1 \otimes B_{21}x_2), \tag{8}$$
$$\Delta_1 x = (T_1 x_1 \otimes B_{22}x_2) - (B_{12}x_1 \otimes T_2 x_2), \tag{9}$$
$$\Delta_2 x = (B_{11}x_1 \otimes T_2 x_2) - (T_1 x_1 \otimes B_{21}x_2). \tag{10}$$

4 Generalized Antieigenvalue and Antieigenvectors of $\mathbb{LTEP}$

Unless sated, otherwise the $\mathbb{LTEP}$ is considered as Right definite and the operator matrices Δ_i are positive definite, $\forall\, i = 1, 2$. Then the operator matrices Δ_i are also nonsingular for $\forall\, i = 0, 1, 2$. The parameters $\nu(\Gamma_i)$, $i = 1, 2$ is defined in [2] for the joint $\mathbb{GEP}$s of the form (5) by extending the idea of [31] as follows,

$$\nu(\Gamma_i) = \min\left\{\frac{Re\,\langle \Delta_i x, \Delta_0 x\rangle}{\|\Delta_i x\|\,\|\Delta_0 x\|} : x \in H, \quad \Delta_i x \neq 0, \quad \Delta_0 x \neq 0\right\}. \tag{11}$$

The problem is to find the pair $(\nu(\Gamma_1), \nu(\Gamma_2))$ for the system (5) and is called generalized antieigenvalue pair. Generalised antieigenvectors corresponding to the pair $(\nu(\Gamma_1), \nu(\Gamma_2))$ are vectors x for which the minimum of (11) are obtained. The standard results of generalized antieigenvalue pair of $\mathbb{LTEP}$ are reported in [2] and computation for right definite case are reported in [3]. For inner products $\langle \Delta_i x, \Delta_0 x\rangle$ involved in (11), the following representation are also possible,

$$\langle \Delta_i x, \Delta_0 x\rangle = \left\langle \Delta_i \Delta_0^{-1} y, y\right\rangle; \qquad i = 1, 2,$$

where $\Delta_0 x = y$. Denote,

$$G_i = \Delta_i \Delta_0^{-1}, \qquad i = 1, 2. \tag{12}$$

Then, the parameter $\nu(\Gamma_i)$ defined in (11) reduces to,

$$\nu(\Gamma_i) = \min\left\{\frac{Re\,\langle G_i y, y\rangle}{\|G_i y\|\,\|y\|} : y \in H, \quad G_i y \neq 0, \quad y \neq 0\right\}, \tag{13}$$
$$\nu(\Gamma_i) = \min_{0 \neq y \in H, G_i y \neq 0}\left\{\frac{Re\,\langle G_i y, y\rangle}{\|G_i y\|\,\|y\|}\right\}. \tag{14}$$

As per our assumptions of (2) the matrices present in the system of (2) are of dimension N, and therefore the order of Δ_i becomes $N \times N$. The increase in the size of the structure of Δ_i makes it computationally challenging to analyze and calculate the pairs $(\nu(\Gamma_1), \nu(\Gamma_2))$ if the matrices are of higher dimension. Again, Gustafson proved that for any accretive operator K, the following equality holds,

$$\sin K = \sqrt{1 - \cos^2 K} = \inf_{\epsilon > 0} \|\epsilon K - I\|\,. \tag{15}$$

Lemma 4.1. *Let $\Delta_0 x$ be as given in (8) and $x = x_1 \otimes x_2$, then,*

$$\|\Delta_0 x\| \leq \|B_{11}\|\,\|x_1\| + \|B_{22}\|\,\|x_2\| + \|B_{12}\|\,\|x_1\| + \|B_{21}\|\,\|x_2\|\,.$$

Proof. Taking norm on both sides of (8) we have,

$$\begin{aligned}\|\Delta_0 x\| &= \|(B_{11}x_1 \otimes B_{22}x_2) - (B_{12}x_1 \otimes B_{21})x_2\|\\ &\leq \|B_{11}x_1 \otimes B_{22}x_2\| + \|B_{12}x_1 \otimes B_{21}x_2\|\\ &\leq \|B_{11}x_1\| + \|B_{22}x_2\| + \|B_{12}x_1\| + \|B_{21}x_2\|\\ &\leq \|B_{11}\|\,\|x_1\| + \|B_{22}\|\,\|x_2\| + \|B_{12}\|\,\|x_1\| + \|B_{21}\|\,\|x_2\|,\end{aligned}$$

which proves the lemma. □

Theorem 4.1. *Let G_i; $i = 1, 2$ be as defined in (12). Then,*

$$\begin{aligned}\sin(G_1) &\leq \left\|\Delta_0^{-1}\right\| \Big[\sin(T_1)\,\|B_{22}\| + \sin(T_2)\,\|B_{12}\| + \sin(B_{11})\,\|B_{22}\| + \sin(B_{12})\,\|B_{21}\|\Big],\\ \sin(G_2) &\leq \left\|\Delta_0^{-1}\right\| \Big[\sin(B_{11})\,\|T_2\| + \sin(T_1)\,\|B_{21}\| + \sin(B_{11})\,\|B_{22}\| + \sin(B_{12})\,\|B_{21}\|\Big].\end{aligned}$$

Proof. It is well known that,

$$\inf(a_n.b_n) \leq \sup(a_n).\inf(b_n). \tag{16}$$

We have,

$$\begin{aligned}&\|\epsilon G_1 - I\|\\ &\quad= \left\|\epsilon\Delta_1\Delta_0^{-1} - I\right\|\\ &\quad= \left\|\epsilon\Delta_1\Delta_0^{-1} - \Delta_0\Delta_0^{-1}\right\|\\ &\quad= \left\|(\epsilon\Delta_1 - \Delta_0)\Delta_0^{-1}\right\|\\ &\quad\leq \left\|\Delta_0^{-1}\right\| \|\epsilon\Delta_1 - \Delta_0\|\\ &\quad= \left\|\Delta_0^{-1}\right\| \|\epsilon T_1 \otimes B_{22} - \epsilon T_2 \otimes B_{12} - B_{11} \otimes B_{22} + B_{12} \otimes B_{21}\|\\ &\quad= \left\|\Delta_0^{-1}\right\| \|(\epsilon T_1 - I) \otimes B_{22} - (\epsilon T_2 - I) \otimes B_{12} + I \otimes B_{22} - B_{11} \otimes B_{22} - I \otimes B_{12} + B_{12} \otimes B_{21}\|\\ &\quad= \left\|\Delta_0^{-1}\right\| \|(\epsilon T_1 - I) \otimes B_{22} - (\epsilon T_2 - I) \otimes B_{12} + (I - B_{11}) \otimes B_{22} - (I - B_{12}) \otimes B_{21}\|\\ &\quad\leq \left\|\Delta_0^{-1}\right\| \Big[\|(\epsilon T_1 - I) \otimes B_{22}\| + \|(\epsilon T_2 - I) \otimes B_{12}\| + \|(I - B_{11}) \otimes B_{22}\| + \|(I - B_{12}) \otimes B_{21}\|\Big]\\ &\quad= \left\|\Delta_0^{-1}\right\| \Big[\|(\epsilon T_1 - I)\|\,\|B_{22}\| + \|(\epsilon T_2 - I)\|\,\|B_{12}\| + \|(I - B_{11})\|\,\|B_{22}\| + \|(I - B_{12})\|\,\|B_{21}\|\Big].\end{aligned}$$

Taking infrimum on both sides when $\epsilon > 0$ we have,

$$\begin{aligned}\implies \inf_{\epsilon>0} \|\epsilon G_1 - I\| &\leq \sup_{\epsilon>0} \left\|\Delta_0^{-1}\right\| \Big[\inf_{\epsilon>0} \|(\epsilon T_1 - I)\|\,\|B_{22}\| + \|(\epsilon T_2 - I)\|\,\|B_{12}\| + \|(I - B_{11})\|\,\|B_{22}\|\\ &\qquad + \|(I - B_{12})\|\,\|B_{21}\|\Big]\\ &= \sup_{\epsilon>0} \left\|\Delta_0^{-1}\right\| [\inf_{\epsilon>0}(\|(\epsilon T_1 - I)\|\,\|B_{22}\|) + \inf_{\epsilon>0}(\|(\epsilon T_2 - I)\|\,\|B_{12}\|)\\ &\qquad + \inf_{\epsilon>0}(\|(I - B_{11})\|\,\|B_{22}\|) + \inf_{\epsilon>0}(\|(I - B_{12})\|\,\|B_{21}\|)]\\ &\leq \sup_{\epsilon>0} \left\|\Delta_0^{-1}\right\| \Big[\sin(T_1) \sup_{\epsilon>0} \|B_{22}\| + \sin(T_2) \sup_{\epsilon>0} \|B_{12}\| + \sin(B_{11}) \sup_{\epsilon>0} \|B_{22}\|\\ &\qquad + \sin(B_{12}) \sup_{\epsilon>0} \|B_{21}\|\Big],\\ \implies \sin(G_1) &\leq \left\|\Delta_0^{-1}\right\| \Big[\sin(T_1)\,\|B_{22}\| + \sin(T_2)\,\|B_{12}\| + \sin(B_{11})\,\|B_{22}\| + \sin(B_{12})\,\|B_{21}\|\Big].\end{aligned}$$

Also,

$$\begin{aligned}
&\|\epsilon G_2 - I\| \\
&= \left\|\epsilon \Delta_2 \Delta_0^{-1} - I\right\| \\
&= \left\|\epsilon \Delta_2 \Delta_0^{-1} - \Delta_0^{-1}\Delta_0\right\| \\
&= \left\|\Delta_0^{-1}(\epsilon \Delta_2 - \Delta_0)\right\| \\
&\leq \left\|\Delta_0^{-1}\right\| \|\epsilon \Delta_2 - \Delta_0\| \\
&= \left\|\Delta_0^{-1}\right\| \|\epsilon B_{11} \otimes T_2 - \epsilon T_1 \otimes B_{21} - B_{11} \otimes B_{22} + B_{12} \otimes B_{21}\| \\
&= \left\|\Delta_0^{-1}\right\| ||B_{11} \otimes (\epsilon T_2 - I) - (\epsilon T_1 - I) \otimes B_{21} + B_{11} \otimes I - I \otimes B_{21} - B_{11} \otimes B_{22} + B_{12} \otimes B_{21}|| \\
&= \left\|\Delta_0^{-1}\right\| ||B_{11} \otimes (\epsilon T_2 - I) - (\epsilon T_1 - I) \otimes B_{21} - B_{11} \otimes (B_{22} - I) + (B_{12} - I) \otimes B_{21}|| \\
&\leq \left\|\Delta_0^{-1}\right\| \Big[\|B_{11} \otimes (\epsilon T_2 - I)\| + \|(\epsilon T_1 - I) \otimes B_{21}\| + \|B_{11} \otimes (B_{22} - I)\| + \|(B_{12} - I) \otimes B_{21}\| \Big] \\
&= \left\|\Delta_0^{-1}\right\| \Big[\|B_{11}\| \|(\epsilon T_2 - I)\| + \|(\epsilon T_1 - I)\| \|B_{21}\| + \|B_{11}\| \|(B_{22} - I)\| + \|(B_{12} - I)\| \|B_{21}\| \Big].
\end{aligned}$$

Taking infrimum on both sides when $\epsilon > 0$ we have,

$$\begin{aligned}
\inf_{\epsilon>0} \|\epsilon G_2 - I\| \leq &\sup_{\epsilon>0} \left\|\Delta_0^{-1}\right\| \inf_{\epsilon>0} \Big[\|B_{11}\| \|(\epsilon T_2 - I)\| + \|(\epsilon T_1 - I)\| \|B_{21}\| + \|B_{11}\| \|(B_{22} - I)\| \\
&+ \|(B_{12} - I)\| \|B_{21}\| \Big] \\
= &\sup_{\epsilon>0} \left\|\Delta_0^{-1}\right\| \Big[\inf_{\epsilon>0}(\|B_{11}\| \|(\epsilon T_2 - I)\|) + \inf_{\epsilon>0}(\|(\epsilon T_1 - I)\| \|B_{21}\|) \\
&+ \inf_{\epsilon>0} \|B_{11}\| (\|(B_{22} - I)\| + \|(B_{12} - I)\| \|B_{21}\|) \Big], \\
\implies \sin(G_2) \leq &\left\|\Delta_0^{-1}\right\| \Big[\|B_{11}\| \sin(T_2) + \sin(T_1) \|B_{21}\| + \|B_{11}\| \sin(B_{22}) + \sin(B_{12}) \|B_{21}\| \Big].
\end{aligned}$$

□

Theorem 4.2. *Let $\nu(\Gamma_i)$; $i = 1, 2$ be as given in (14), then,*

$$\nu(\Gamma_i) \leq \sup_{\|y\|=1} Re\, \langle G_i y, y\rangle . \inf \frac{1}{\|G_i\|}.$$

Proof. Recall (16) and using it in the expansion of $\nu(\Gamma_i)$, we have,

$$\begin{aligned}
\nu(\Gamma_i) = \inf_{\|y\|=1} \frac{Re\, \langle G_i y, y\rangle}{\|G_i y\| \|y\|} &\leq \sup_{\|y\|=1} Re\, \langle G_i y, y\rangle . \inf_{\|y\|=1} \frac{1}{\|G_i y\|} \\
&\leq \sup_{\|y\|=1} Re\, \langle G_i y, y\rangle . \inf_{\|y\|=1} \frac{1}{\|G_i\|} \qquad (17) \\
&= \sup_{\|y\|=1} y^T G_i y . \inf \frac{1}{\|G_i\|}.
\end{aligned}$$

□

5 Calculation of $\nu(\Gamma_i)$

Here we adopt a direct procedure to compute the generalized antieigenvalue pair as well as its corresponding generalized antieigenvectors defined in (11). For $i = 1, 2$, the matrices G_i are real, symmetric, and positive definite. So, immediate consequences of the Cauchy Schwarz inequality for G_i are given by,

$$(y^T G_i y)^2 \leq y^T G_i^2 y . y^T y.$$

The equality holds if and only if $G_i y$ is proportional to y. Thus, for $0 \neq y$ we have,

$$\frac{y^T G_i y}{\sqrt{y^T G_i^2 y . y^T y}} \leq 1.$$

In other words, this can be restated as the optimization problems,

$$\max_{y \neq 0} \frac{y^T G_i y}{\sqrt{y^T G_i^2 y . y^T y}},$$

have their optimal value 1 and the solutions to the above optimization problems are given by the set of all eigenvectors of the matrices G_i. Moreover, the corresponding minimization problems,

$$\min_{y \neq 0} \frac{y^T G_i y}{\sqrt{y^T G_i^2 y . y^T y}},$$

yield a lower bound on $\frac{y^T G_i y}{\sqrt{y^T G_i^2 y . y^T y}}$. Inspired by this work, the pairs of generalized antieigenvalues and their corresponding generalized antieigenvectors of G_i can be obtained as the general solutions of their associated optimization problems,

$$\nu(\Gamma_i) = \min_{y \neq 0} \frac{y^T G_i y}{\sqrt{y^T G_i^2 y . y^T y}}.$$

Let us consider, the optimization problems with the following functions,

$$h(y) = \frac{y^T G_i y}{\sqrt{y^T G_i^2 y . y^T y}}. \tag{18}$$

The problems are to evaluate the stationary values of (18). Taking logarithm on both sides of (18) yields the equation below,

$$\log[h(y)] = \log(y^T G_i y) - \frac{1}{2}\log(y^T G_i^2 y) - \frac{1}{2}\log(y^T y). \tag{19}$$

Taking matrix derivative on both sides of (19) with respect to y we get,

$$\frac{1}{h(y)}\frac{\partial h(y)}{\partial y} = \frac{1}{y^T G_i y}\frac{\partial (y^T G_i y)}{\partial y} - \frac{1}{2}\frac{\partial (y^T G_i^2 y)}{\partial y} - \frac{1}{2}\frac{1}{y^T y}\frac{\partial (y^T y)}{\partial y},$$
$$\implies \frac{\partial h(y)}{\partial y} = h(y)\left[\frac{1}{y^T G_i y}\frac{\partial (y^T G_i y)}{\partial y} - \frac{1}{2}\frac{1}{y^T G_i^2 y}\frac{\partial (y^T G_i^2 y)}{\partial y} - \frac{1}{2}\frac{1}{y^T y}\frac{\partial (y^T y)}{\partial y}\right].$$

Since each $G_i, i = 1, 2$ is symmetric and therefore by Lemma 2.2 we have,

$$\frac{\partial h(y)}{\partial y} = h(y)\left[\frac{1}{y^T G_i y} 2G_i y - \frac{1}{2y^T G_i^2 y} 2G_i^2 y - \frac{1}{2y^T y} 2y\right].$$

After equating to zero, resulting equation of matrix derivative of $h(y)$ for $i = 1$ w.r.t. y becomes,

$$\frac{1}{y^T G_1 y} 2G_1 y - \frac{1}{y^T G_1^2 y} G_1^2 y - \frac{1}{y^T y} = 0. \tag{20}$$

Pre-multiplying both sides of (20) by $y^T G_1 y$ we have,

$$\frac{y^T G_1 y}{y^T G_1^2 y}.G_1^2 y + \frac{y^T G_1 y}{y^T y}.y = 2G_1 y. \tag{21}$$

Then, an orthogonal matrix P and a diagonal matrix D exist such that $G_1 = PDP^T$, where $D = dia(\lambda_1, \lambda_2, \ldots, \lambda_k)$. Define $z = P^T y$, then (18) reduces to,

$$h(z) = \frac{z^T D z}{\sqrt{z' D^2 z . z' z}}, \tag{22}$$

and (21) can be expressed as,

$$\begin{aligned} \gamma_1 G_1^2 P z + \gamma_2 P z &= 2G_1 P z, \\ \gamma_1 z^T P^T G_1^2 P z &= z^T P^T G_1 P z, \end{aligned} \tag{23}$$

or

$$\begin{aligned} 2Dz &= \gamma_1 D^2 z + \gamma_2 z, \\ z^T D z &= \gamma_1 z^T D^2 z. \end{aligned} \tag{24}$$

There are p individual ways to write the matrix (24) as follows:

$$2\lambda_i z_i = (\gamma_1 \lambda_i^2 + \gamma_2) z_i; \qquad i := 1 : p. \tag{25}$$

The following two cases can be considered when solving (24) and (25),

Case 1: The vector $z = e_i$, where e_i is the i^{th} column of identity matrix of order p, is a solution of the equations (24) because,

$$\begin{aligned} 2De_i &= \gamma_1 D^2 e_i + \gamma_2 e_i, \\ e_i^T D e_i &= \gamma_1 e_i^{'} D^2 e_i, \end{aligned} \tag{26}$$

with $\gamma_1 = \lambda_i^{-1}$ and $\gamma_2 = \lambda_i$. As a result, the stationary value $h(e_i) = 1$, which corresponds to the maximum value of $h(y)$. In this situation, the eigenvectors are undoubtedly the choice of vectors that maximise $h(y)$.

Case 2: The vectors of the form $z = d_i e_i + d_j e_j; d_i \neq 0, d_j \neq 0$ are potential candidates. After replacing z in (24) with this selection, we get,

$$\begin{aligned} 2\lambda_i &= \gamma_1 \lambda_i^2 + \gamma_2, \\ 2\lambda_j &= \gamma_1 \lambda_j^2 + \gamma_2, \end{aligned} \tag{27}$$

with choice $\lambda_i \neq \lambda_j$. This results in $\gamma_1 = 2(\lambda_i + \lambda_j)^{-1}$ and $\gamma_2 = 2\lambda_i \lambda_j (\lambda_i + \lambda_j)^{-1}$.

However, it is observed that $\frac{\gamma_2}{\gamma_1} = \frac{z^T D^2 z}{z^T z}$ and thus a solution of the form $z = d_i e_i + d_j e_j$ will exist only if $\lambda_i^2 < \frac{\gamma_2}{\gamma_1} < \lambda_j^2;\ j > i$. Furthermore, in this instance, the equation specified in (24) necessitates that,

$$\left(\frac{d_i}{d_j}\right)^2 = \frac{\lambda_j}{\lambda_i}. \tag{28}$$

The corresponding stationary values are equal to,

$$\sqrt{\gamma_1\gamma_2} = \frac{2\sqrt{\lambda_i\lambda_j}}{\lambda_i + \lambda_j}. \tag{29}$$

Thus, using (28), we have $\sqrt{\lambda_j}e_i \pm \sqrt{\lambda_i}e_j,\ \lambda_i \neq \lambda_j$ as the solutions to (24). Obviously, any solution to (25) will fall into two categories, either in Case 1 or in Case 2. Thus, all potential options of the generalized antieigenvectors and the corresponding generalized antieigenvalues can be listed as those given by Case 2, written as a set,

$$R = \left\{ \sqrt{\lambda_j}e_i + \sqrt{\lambda_i}e_j, \quad \lambda_i \neq \lambda_j, \quad i, j = 1 : p \right\}.$$

This list might be quite lengthy. But it follows from the previous Lemma 2.1 that $\dfrac{2\sqrt{\lambda_1\lambda_p}}{\lambda_1 + \lambda_p}$ is the first generalized antieigenvalue with corresponding vectors z_1 and z_2 as $(\sqrt{\lambda_1}, 0, \ldots, 0, \pm\sqrt{\lambda_p})^T$ and corresponding generalized antieigenvectors as $y_1 = Pz_1$ and $y_2 = Pz_2$. It is to be noted that these vectors are not mutually orthogonal, unless $\lambda_1 = \lambda_p$.

By inserting the corresponding nonzero entries in the appropriate locations, pairs that are similar to z_1 and z_2 can be obtained if either λ_1 and λ_p has repeating roots. Proceeding in this way, and using the Lemma 2.1 and the fact that for every z, there will be precisely two nonzero components and assuming that λ_1 and λ_p are not occur repeatedly as a roots of (18), the next generalized antieigenvector which is orthogonal to z_1 and z_2 becomes,

$$\left(0, \sqrt{\lambda_2}, 0, \ldots\ldots, 0, \pm\sqrt{\lambda_{p-1}}, 0\right)^T,$$
$$\lambda_1 \neq \lambda_2, \quad \lambda_{p-1} \neq \lambda_p, \quad \lambda_2 \neq \lambda_{p-1}.$$

Thus, the next generalized antieigenvalue will be $\min\left(\dfrac{2\sqrt{\lambda_2\lambda_{p-1}}}{\lambda_2 + \lambda_{p-1}}, 1\right)$. If it is 1, then all subsequent antieigenvalues will be 1, since this is the maximum of (18) can attain, otherwise, the process keeps on the same manner. Evidently, repeated eigenvalues could result in generalized antieigenvectors that are non-orthogonal and relate to the same generalized antieigenvalue.

In a similar manner, considering $i = 2$ in (18), the generalized antieigenvalue $\nu(\Gamma_2)$ can be calculated.

Example 5.1. *Consider the* $\mathbb{LTEP}$ *represented by* (30) *and* (31) *with real symmetric matrices:*

$$L_1(\lambda,\mu)x_1 := \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \lambda\begin{pmatrix} 8 & 2 \\ 2 & 9 \end{pmatrix} - \mu\begin{pmatrix} 1 & 5 \\ 5 & 2 \end{pmatrix}\right] x_1 = 0, \tag{30}$$

$$L_2(\lambda,\mu)x_2 := \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \mu\begin{pmatrix} 6 & -1 \\ -1 & 7 \end{pmatrix}\right] x_2 = 0. \tag{31}$$

Then, the associated operator matrices are,

$$\Delta_0 = \begin{pmatrix} 47 & -8 & 7 & -2 \\ -8 & 55 & -2 & 9 \\ 7 & -2 & 52 & -9 \\ -2 & 9 & -9 & 61 \end{pmatrix}, \quad \Delta_1 = \begin{pmatrix} 5 & -1 & -5 & 0 \\ -1 & 6 & 0 & -5 \\ -5 & 0 & 4 & -1 \\ 0 & -5 & -1 & 5 \end{pmatrix}, \quad \Delta_2 = \begin{pmatrix} 7 & 0 & 2 & 0 \\ 0 & 7 & 0 & 2 \\ 2 & 0 & 8 & 0 \\ 0 & 2 & 0 & 8 \end{pmatrix},$$

$$G_1 = \begin{pmatrix} 0.1225 & -0.0025 & -0.1149 & -0.0126 \\ -0.0025 & 0.1250 & -0.0126 & -0.1023 \\ -0.1226 & -0.0137 & 0.0921 & -0.0048 \\ -0.0137 & -0.1089 & -0.0048 & 0.0969 \end{pmatrix}, G_2 = \begin{pmatrix} 0.1499 & 0.0218 & 0.0199 & 0.0046 \\ 0.0218 & 0.1281 & 0.0046 & 0.0153 \\ 0.0213 & 0.0050 & 0.1551 & 0.0228 \\ 0.0050 & 0.0163 & 0.0228 & 0.1323 \end{pmatrix}.$$

Table 1: Eigenvalue and eigenvectors of real symmetric matrices of Example 5.1.

λ	μ	(λ, μ)	Eigenvectors
-0.0228	$+0.1900$	$(+0.2288, +0.1900)$	$\begin{pmatrix}-0.5731 & -0.3542 & -0.6286 & -0.3885\end{pmatrix}^T$
$+0.0148$	$+0.1035$	$(+0.0148, +0.1293)$	$\begin{pmatrix}-0.3515 & +0.5687 & -0.3909 & +0.6326\end{pmatrix}^T$
$+0.2329$	$+0.1293$	$(+0.2329, +0.1425)$	$\begin{pmatrix}-0.6471 & -0.3999 & +0.5522 & +0.3413\end{pmatrix}^T$
$+0.2116$	$+0.1425$	$(+0.2116, +0.1035)$	$\begin{pmatrix}+0.4013 & -0.6493 & -0.3397 & +0.5496\end{pmatrix}^T$

Eigenvalues and eigenpairs of real symmetric matrices are presented in Table 1. The first generalized antieigenvalue pair is $(0.1047i, 0.9556)$ *with corresponding antieigenvectors are,*

$$\left(\sqrt{0.2329}e_1 \pm i\sqrt{0.0228}e_4\right); \quad \left(\sqrt{0.1900}e_1 \pm \sqrt{0.1035}e_4\right).$$

The second generalized antieigenvalue pair is $(0.4944, 0.9989)$ *with corresponding antieigenvectors are,*

$$\left(\sqrt{0.2116}e_2 \pm i\sqrt{0.0148}e_3\right); \quad \left(\sqrt{0.1425}e_2 \pm \sqrt{0.1293}e_3\right).$$

Example 5.2. *Consider the following* $\mathbb{LTEP}$ *defined in (32) and (33) with real diagonal matrices.*

$$L_1(\lambda,\mu)x_1 := \left[\begin{pmatrix} 9 & 0 \\ 0 & 10 \end{pmatrix} - \lambda\begin{pmatrix} 6 & 0 \\ 0 & 7 \end{pmatrix} - \mu\begin{pmatrix} -5 & 0 \\ 0 & -3 \end{pmatrix}\right] x_1 = 0, \tag{32}$$

$$L_2(\lambda,\mu)x_2 := \left[\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} - \lambda\begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix} - \mu\begin{pmatrix} 4 & 0 \\ 0 & 5 \end{pmatrix}\right] x_2 = 0, \tag{33}$$

$$\Delta_0 = \begin{pmatrix} 39 & 0 & 0 & 0 \\ 0 & 50 & 0 & 0 \\ 0 & 0 & 37 & 0 \\ 0 & 0 & 0 & 47 \end{pmatrix}, \quad \Delta_1 = \begin{pmatrix} 46 & 0 & 0 & 0 \\ 0 & 55 & 0 & 0 \\ 0 & 0 & 46 & 0 \\ 0 & 0 & 0 & 56 \end{pmatrix}, \quad \Delta_2 = \begin{pmatrix} -15 & 0 & 0 & 0 \\ 0 & -24 & 0 & 0 \\ 0 & 0 & -16 & 0 \\ 0 & 0 & 0 & -26 \end{pmatrix},$$

$$G_1 = \begin{pmatrix} 1.1795 & 0 & 0 & 0 \\ 0 & 1.1000 & 0 & 0 \\ 0 & 0 & 1.2432 & 0 \\ 0 & 0 & 0 & 1.1915 \end{pmatrix}, \quad G_2 = \begin{pmatrix} -0.3846 & 0 & 0 & 0 \\ 0 & -0.4800 & 0 & 0 \\ 0 & 0 & -0.4324 & 0 \\ 0 & 0 & 0 & -0.5532 \end{pmatrix}.$$

Table 2: Eigenvalue and eigenvectors of real diagonal matrices of Example 5.2.

λ	μ	(λ, μ)	Eigenvectors
$+1.1000$	-0.5532	$(+1.1795, -0.3846)$	$\begin{pmatrix}1 & 0 & 0 & 0\end{pmatrix}^T$
$+1.1795$	-0.4800	$(+1.1000, -0.4800)$	$\begin{pmatrix}0 & 1 & 0 & 0\end{pmatrix}^T$
$+1.1915$	-0.4324	$(+1.2432, -0.4324)$	$\begin{pmatrix}0 & 0 & 1 & 0\end{pmatrix}^T$
$+1.2432$	-0.3846	$(+1.1915, -0.5532)$	$\begin{pmatrix}0 & 0 & 0 & 1\end{pmatrix}^T$

Eigenvalues of $\mathbb{LTEP}$ *with diagonal matrices are presented in Table 2. The first generalized antieigenvalue pair is* $(0.9981, -2.0653)$ *with corresponding antieigenvectors are,*

$$\left\{\sqrt{1.2432}e_1 \pm \sqrt{1.1000}e_4; \;\; i\sqrt{0.3846}e_1 \pm i\sqrt{0.5532}e_4\right\}.$$

The second generalized antieigenvalue pair is $(1, -0.0692)$ *with corresponding antieigenvectors are,*

$$\left\{\sqrt{1.1915}e_2 \pm \sqrt{1.1795}e_4; \;\; i\sqrt{0.4800}e_2 \pm i\sqrt{0.4324}e_4\right\}.$$

Example 5.3. *Consider the following two-parameter problem defined in (34) and (35) with real positive definite matrices:*

$$\mathbb{L}_1(\lambda,\mu)x_1 := \left[\begin{pmatrix} 10 & 5 & 2 \\ 5 & 3 & 2 \\ 2 & 2 & 3 \end{pmatrix} - \lambda \begin{pmatrix} 2 & -1 & 0 \\ -1 & 5 & -1 \\ 0 & -1 & 7 \end{pmatrix} - \mu \begin{pmatrix} 20 & 4 & 5 \\ 4 & 2 & 3 \\ 5 & 3 & 5 \end{pmatrix}\right] x_1 = 0, \tag{34}$$

$$\mathbb{L}_2(\lambda,\mu)x_2 := \left[\begin{pmatrix} -1 & 2 & 3 \\ 2 & 5 & -3 \\ 3 & -3 & 2 \end{pmatrix} - \lambda \begin{pmatrix} 9 & 2 & 3 \\ 2 & -1 & 8 \\ 3 & -8 & 0 \end{pmatrix} - \mu \begin{pmatrix} 12 & 4 & 3 \\ 4 & 2 & -1 \\ 3 & -1 & 7 \end{pmatrix}\right] x_2 = 0, \tag{35}$$

$$\Delta_0 = \begin{pmatrix} -156 & -32 & -54 & -48 & -12 & -15 & -45 & -10 & -15 \\ -32 & 24 & -162 & -12 & 2 & -31 & -10 & 5 & -40 \\ -54 & 158 & 14 & -15 & 33 & -7 & -15 & 40 & 0 \\ -48 & -12 & -15 & 42 & 16 & 9 & -39 & -10 & -12 \\ -12 & 2 & -31 & 16 & 12 & -21 & -10 & 1 & -23 \\ -15 & 33 & -7 & 9 & 11 & 35 & -12 & 25 & -7 \\ -45 & -10 & -15 & -39 & -10 & -12 & 39 & 18 & 6 \\ -10 & 5 & -40 & -10 & 1 & -23 & 18 & 19 & -47 \\ -15 & 40 & 0 & -12 & 25 & -7 & 6 & 33 & 49 \end{pmatrix},$$

$$\Delta_1 = \begin{pmatrix} 140 & 0 & -30 & 64 & 12 & 3 & 29 & -2 & -9 \\ 0 & -80 & 50 & 12 & -10 & 7 & -2 & -21 & 13 \\ -30 & 50 & 30 & 3 & 7 & 27 & -9 & 13 & 4 \\ 64 & 12 & 3 & 38 & 3 & 27 & 2 & -3 & \\ 12 & -10 & 7 & 8 & -4 & 3 & 2 & -11 & 7 \\ 3 & 7 & 27 & 3 & 3 & 17 & -3 & 7 & 8 \\ 29 & -2 & -9 & 27 & 2 & -3 & 41 & 2 & -6 \\ -2 & -21 & 13 & 2 & -11 & 7 & 2 & -19 & 12 \\ -9 & 13 & 4 & -3 & 7 & 8 & -6 & 12 & 11 \end{pmatrix},$$

$$\Delta_2 = \begin{pmatrix} -92 & -16 & -24 & -44 & -12 & -18 & -18 & -4 & -6 \\ -16 & 20 & -86 & -12 & 0 & -37 & -4 & 2 & -16 \\ -24 & 74 & 4 & -18 & 43 & -2 & -6 & 16 & 0 \\ -44 & -12 & -18 & -32 & 4 & 6 & -17 & -6 & -9 \\ -12 & 0 & -37 & 4 & 28 & -39 & -6 & -3 & -13 \\ -18 & 43 & -2 & 6 & 9 & 10 & -9 & 19 & -2 \\ -18 & -4 & -6 & -17 & -6 & -9 & -34 & 8 & 12 \\ -4 & 2 & -16 & -6 & -3 & -13 & 8 & 38 & -45 \\ -6 & 16 & 0 & -9 & 19 & -2 & 12 & 3 & 14 \end{pmatrix},$$

$$G_1 = \begin{pmatrix} -1.0089 & 0.4512 & -0.2460 & 0.3399 & 0.1308 & -0.0240 & -0.0266 & -0.0163 & 0.0047 \\ 0.1499 & -0.4663 & -0.3936 & 0.0388 & 0.5693 & 0.0106 & 0.1378 & -0.2110 & -0.0105 \\ -0.0809 & -0.0274 & 0.3814 & 0.7115 & -1.3370 & 0.0326 & 0.0456 & 0.3502 & -0.0838 \\ -0.7480 & 0.2079 & 0.0707 & 0.8193 & -0.6685 & -0.3717 & 0.3291 & 0.2699 & -0.0681 \\ -0.1436 & 0.0899 & 0.0126 & 0.3574 & -0.5502 & -0.3183 & 0.0455 & -0.0098 & -0.0588 \\ 0.0298 & -0.1799 & -0.0146 & -0.1344 & 0.1455 & 0.3788 & -0.0254 & -0.1110 & 0.0117 \\ -0.7567 & 0.1748 & 0.0815 & 0.6452 & 0.4120 & -0.1112 & 1.2440 & -0.4410 & -0.4513 \\ -0.0757 & 0.0392 & 0.0142 & 0.0618 & 0.2002 & -0.0797 & 0.4565 & -0.7123 & -0.3877 \\ 0.0760 & -0.0153 & -0.0145 & 0.1718 & -0.6236 & 0.0493 & -0.2769 & 0.3685 & 0.3790 \end{pmatrix},$$

$$G_2 = \begin{pmatrix} 0.6085 & 0.0813 & 0.0961 & 0.7037 & -2.2361 & -0.9162 & -0.5255 & 1.3093 & 0.4423 \\ -0.2581 & 1.2213 & 0.3251 & 4.5357 & -10.8647 & -4.1632 & -2.6485 & 6.1067 & 2.1896 \\ 1.9622 & -4.3897 & -1.1163 & -28.9517 & 68.3516 & 23.3954 & 15.4873 & -37.3789 & -12.3970 \\ 1.5924 & -2.5304 & -0.9610 & -17.0857 & 39.4291 & 13.7966 & 8.7621 & -21.6388 & -7.2960 \\ 1.0149 & -2.1849 & -0.7883 & -14.6185 & 35.5247 & 11.3870 & 7.7640 & -18.8263 & -6.0252 \\ -0.3488 & 0.7440 & 0.4486 & 4.9951 & -11.5766 & -3.6175 & -2.6395 & 6.3864 & 2.1813 \\ -0.2273 & 1.8691 & 0.4890 & 12.6307 & -31.4740 & -10.6568 & -8.3637 & 17.7151 & 6.5146 \\ -0.6414 & 1.6867 & 0.5035 & 12.9688 & -31.3651 & -10.2941 & -7.7666 & 18.7697 & 6.1998 \\ 0.9540 & -2.4356 & -0.7718 & -16.4004 & 39.1445 & 13.2599 & 9.3083 & -21.6918 & -7.1049 \end{pmatrix}.$$

Table 3: Eigenvalue pair of Example 5.3.

λ	μ	(λ, μ)
$+1.5698$	$+20.1443$	$(+1.5698, +20.1443)$
-1.0487	-3.5247	$(-1.0487, -3.5247)$
-0.5476	-1.3883	$(-0.5476, -1.3883)$
-0.5476	$+1.3189$	$(-0.5476, +1.3189)$
$+0.2605$	$+0.3530 + 0.1238i$	$(+0.2605, 0.3530 + 0.1238i)$
$+0.1893$	$+0.3530 - 0.1238i$	$(+0.1893, 0.3530 - 0.1238i)$
$-0.1609 + 0.1653i$	$+0.2545$	$(-0.1609 + 0.1653i, +0.2545)$
$-0.1609 - 0.1653i$	$+0.5827$	$(-0.1609 - 0.1653i, +0.5827)$
-0.1705	$+0.7430$	$(-0.1705, +0.7430)$

Eigenvalue pairs of the $\mathbb{LTEP}$ *in Example 5.3 are shown in Table 3.*

- *The first generalised antieigenvalue pair is* $(+4.9244i, +1.0140i)$ *with corresponding antieigenvectors* $(\sqrt{1.5698}\, e_1 \pm i\, \sqrt{1.0487}\, e_9; \sqrt{20.1443}\, e_1 \pm i\, \sqrt{3.5247}\, e_9)$.
- *The second generalised antieigenvalue pair is* $(-78.9353i, -38.9958i)$ *with corresponding antieigenvectors* $(\sqrt{0.5339}\, e_2 \pm i\, \sqrt{0.5476}\, e_8; \sqrt{1.3189}\, e_2 \pm i\, \sqrt{1.3883}\, e_8)$.
- *The third generalised antieigenvalue pair is* $(+4.6833i, +0.8719)$ *with corresponding antieigenvectors* $(\sqrt{0.2605}\, e_3 \pm i\, \sqrt{0.1705}\, e_7; \sqrt{0.7430}\, e_3 \pm \sqrt{0.2545}\, e_7)$.
- *The fourth generalised antieigenvalue pair is* $(2.4266 - 0.5664\, i, 0.9886 + i\, 0.0367)$ *with corresponding antieigenvectors* $\left(\sqrt{-0.1609 + i\, 0.1653}\, e_4 \pm \sqrt{0.1893}\, e_6; \sqrt{0.5827}\, e_4 \pm \sqrt{0.3530 + i\, 0.1238}\, e_6\right)$.
- *The fifth generalised antieigenvalue pair is* $(2.4266 + i\, 0.5664, 0.9886 - i\, 0.0367)$ *with corresponding antieigenvectors* $(\sqrt{-0.1609 - i\, 0.1653}\, e_4 \pm \sqrt{0.1893}\, e_5; \sqrt{0.5827}\, e_4 \pm \sqrt{0.3530 - i\, 0.1238}\, e_5)$.

All calculations are performed in the environment MATLAB $R2019a$ with Windows 11 operating system, AMD Ryzen 5 5500U 2.10 GHz processor and using the package MultiParEig available in [33]. Randomly generated matrices of order n can be considered by using the MATLAB command *randn*(n). However, the dimension of the associated joint $\mathbb{GEP}$ of the $\mathbb{LTEP}$ of coefficient matrices of order n increases to n^2. Therefore, the computation of generalised antieigenvalue pair will be complex tasks for large order matrices. The generalized antieigenvalue pairs can also be computed by solving the optimization problems generated from (11) directly. To solve the optimization problem, the quantity $x^T \Delta_i^T \Delta_0 x$ appears explicitly for $i := 1, 2$. These can further be spitted to get new expressions involving Kronecker product of coefficient matrices of $\mathbb{LTEP}$ are

are given by,

$$\begin{aligned}\Delta_1^T\Delta_0 =&(T_1\otimes B_{22}-B_{12}\otimes T_2)^T(B_{11}\otimes B_{22}-B_{12}\otimes B_{21})\\ =&(T_1^T\otimes B_{22}^T-B_{12}^T\otimes T_2^T)(B_{11}\otimes B_{22})-(T_1^T\otimes B_{22}^T-B_{12}^T\otimes T_2^T)(B_{12}\otimes B_{21})\\ =&(T_1^T\otimes B_{22}^T)(B_{11}\otimes B_{22})-(B_{12}^T\otimes T_2^T)(B_{11}\otimes B_{22})-(T_1^T\otimes B_{22}^T)(B_{12}\otimes B_{21})\\ &+(B_{12}^T\otimes T_2^T)(B_{12}\otimes B_{21})\\ =&(T_1^TB_{11}\otimes B_{22}^TB_{22})-(B_{12}^TB_{11}\otimes T_2^TB_{22})-(T_1^TB_{12}\otimes B_{22}^TB_{21})+(B_{12}^TB_{12}\otimes T_2^TB_{21}),\end{aligned} \tag{36}$$

which implies,

$$\begin{aligned}x^T\Delta_1^T\Delta_0x =&(x_1^TT_1^TB_{11}x_1)(x_2^TB_{22}^TB_{22}x_2)-(x_1^TB_{12}^TB_{11}x_1)(x_2^TT_2^TB_{22}x_2)\\ &-(x_1^TT_1^TB_{12}x_1)(x_2^TB_{22}^TB_{21}x_2)+(x_1^TB_{12}^TB_{12}x_1)(x_2^TT_2^TB_{21}x_2).\end{aligned} \tag{37}$$

Similar expressions for $x^T\Delta_2^T\Delta_0x$, $x^T\Delta_1^T\Delta_1x$ and $x^T\Delta_0^T\Delta_0x$ can also be derived. Using these values, we may arrive to the following remark.

Remark 5.1. *The generalized antieigenvalue pairs and the corresponding generalized antieigenvectors of* $\mathbb{LTEP}$ *are the solution of the optimization problems with the following functions,*

$$\frac{x^T\Delta_i^T\Delta_0x}{\sqrt{x^T\Delta_i^T\Delta_ix.x^T\Delta_0^T\Delta_0x}};\qquad i=1,2. \tag{38}$$

6 Conclusion

We discussed the abstract algebraic setting of $\mathbb{LTEP}$ and their generalized antieigenvalue theory. We computed generalized antieigenvalue pair of Right definite $\mathbb{LTEP}$ by solving the relevant optimization problems. To find generalised antieigenvalue pair from (11) is computationally challenging, if the operator determinant Δ_0 singular and the coefficient matrices are of larger dimension. Therefore, a different approach is required to study the generalized antieigenvalue theory for singular $\mathbb{LTEP}$, and it can be considered as future avenue of further research in this area. It is also anticipated that further research is necessary for deeper understanding on the connections between generalized antieigenvalue pairs and other concepts in the spectral theory of $\mathbb{LTEP}$ such as eigenvalues and singular values. Moreover, exploring new applications of generalized antieigenvalue pairs of $\mathbb{LTEP}$ can contribute to the advancements of diverse scientific domains.

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Conflicts of Interest All authors declare that there is no conflict of interest in the whole work.

References

[1] F. V. Atkinson & A. B. Mingarelli (1972). *Multiparameter eigenvalue problems* volume 1. Academic Press, New York.

[2] N. Bora (2020). On generalized antieigenvalue pair and antieigenvectors of linear two-parameter matrix eigenvalue problems. *International Journal of Applied Mathematics and Statistics*, *59*(3), 93–101.

[3] N. Bora (2021). Computation of generalized antieigenvalue pair of right definite two-parameter eigenvalue problems. *Journal of Scientific Research*, *65*(5), 208–211. https://doi.org/10.37398/JSR.2021.650524.

[4] N. Cottin (2001). Dynamic model updating – a multiparameter eigenvalue problem. *Mechanical systems and signal processing*, *15*(4), 649–665. https://doi.org/10.1006/mssp.2000.1362.

[5] K. De Cock & B. De Moor (2021). Multiparameter eigenvalue problems and shift-invariance. *IFAC-PapersOnLine*, *54*(9), 159–165. https://doi.org/10.1016/j.ifacol.2021.06.071.

[6] H. Z. F. Ding (2013). On the Kronecker products and their applications. *Journal of Applied Mathematics*, pp. Article ID: 296185. https://doi.org/10.1155/2013/296185.

[7] H. Eisenmann & Y. Nakatsukasa (2022). Solving two-parameter eigenvalue problems using an alternating method. *Linear Algebra and its Applications*, *643*, 137–160. https://doi.org/10.1016/j.laa.2022.02.024.

[8] G. H. Golub & C. F. V. Loan (1983). *Matrix Computations*. Johns Hopkins University Press, Baltimore.

[9] C. Guo, M. Jin, Q. Guo & Y. Li (2018). Antieigenvalue-based spectrum sensing for cognitive radio. *IEEE Wireless Communications Letters*, *8*(2), 544–547. https://doi.org/10.1109/LWC.2018.2879339.

[10] K. Gustafson (1968). The angle of an operator and positive operator products. *Bulletin of American Mathematical Society*, *74*(3), 448–492. https://doi.org/10.1090/S0002-9904-1968-11974-3.

[11] K. Gustafson (1968). Positive (noncommuting) operator products and semigroups. *Mathematische Zeitschrift*, *105*(2), 160–172. https://doi.org/10.1007/BF01110442.

[12] K. Gustafson (2002). Operator trigonometry of statistics and econometrics. *Linear Algebra and Its Applications*, *354*(1-3), 141–158. https://doi.org/10.1016/S0024-3795(01)00315-9.

[13] K. Gustafson (2004). Interaction antieigenvalues. *Journal of Mathematical Analysis and Applications*, *299*(1), 174–185. https://doi.org/10.1016/j.jmaa.2004.06.012.

[14] K. Gustafson (2012). *Antieigenvalue Analysis. With Applications to Numerical Analysis, Wavelets, Statistics, Quantum Mechanics, Finance and Optimization*. World Scientific Publishing Co. Pte. Ltd, Singapore.

[15] K. Gustafson (2016). Antieigenvalue analysis for continuum mechanics, economics, and number theory. *Special Matrices*, *4*(1), 1–8. https://doi.org/10.1515/spma-2016-0001.

[16] K. Gustafson & M. Seddighin (1989). Antieigenvalue bounds. *Journal of Mathematical Analysis and Applications*, *143*(2), 327–340. https://doi.org/10.1016/0022-247X(89)90044-9.

[17] K. Gustafson & M. Seddighin (1993). A note on total antieigenvectors. *Journal of Mathematical Analysis and Applications*, *178*(2), 603–611. https://doi.org/10.1006/jmaa.1993.1329.

[18] M. E. Hochstenbach, T. Košir & B. Plestenjak (2024). Numerical methods for rectangular multiparameter eigenvalue problems, with applications to finding optimal ARMA and LTI models. *Numerical Linear Algebra with Applications*, *31*(2), Article ID: e2540. https://doi.org/10.1002/nla.2540.

[19] M. E. Hochstenbach & B. Plestenjak (2003). Backward error, condition numbers, and pseudospectra for the multiparameter eigenvalue problem. *Linear Algebra and Its Applications*, *375*, 63–81. https://doi.org/10.1016/S0024-3795(03)00613-X.

[20] S. M. Hossein, K. Paul, L. Debnath & K. C. Das (2008). Symmetric anti-eigenvalue and symmetric anti-eigenvector. *Journal of Mathematical Analysis and Applications*, *345*(2), 771–776. https://doi.org/10.1016/j.jmaa.2008.04.061.

[21] R. Khattree (2001). On the calculation of antieigenvalues and antieigenvectors. *Journal of Interdisciplinary Mathematics*, *4*(2-3), 195–199. https://doi.org/10.1080/09720502.2001.10700300.

[22] R. Khattree (2002). On generalized antieigenvalue and antieigenmatrix of order r. *American Journal of Mathematical and Management Sciences*, *22*(1-2), 89–98. https://doi.org/10.1080/01966324.2002.10737577.

[23] R. Khattree (2003). Antieigenvalues and antieigenvectors in statistics. *Journal of Statistical Planning and Inference*, *114*(1-2), 131–144. https://doi.org/10.1016/S0378-3758(02)00467-6.

[24] T. Košir (1994). Finite-dimensional multiparameter spectral theory: the nonderogatory case. *Linear Algebra and Its Applications*, *212*, 45–70. https://doi.org/10.1016/0024-3795(94)90396-4.

[25] T. Košir & B. Plestenjak (2022). On the singular two-parameter eigenvalue problem ii. *Linear Algebra and Its Applications*, *649*, 433–451. https://doi.org/10.1016/j.laa.2022.05.013.

[26] M. G. Krein (1969). Angular localization of the spectrum of a multiplicative integral in a Hilbert space. *Functional Analysis and Its Applications*, *3*(1), 73–74. https://doi.org/10.1007/BF01078278.

[27] B. Mirman (1983). Antieigenvalues: method of estimation and calculation. *Linear Algebra and Its Applications*, *49*, 247–255. https://doi.org/10.1016/0024-3795(83)90107-6.

[28] Z. Muminov & F. Ismail (2016). Threshold resonances and eigenvalues of some schrödinger operators on lattices. *Malaysian Journal of Mathematical Sciences*, *10*(S), 389–407.

[29] M. M. S. Nasser (2019). The explicit eigenvalues and eigenfunctions of the generalized Neumann kernel in annular domains. *Malaysian Journal of Mathematical Sciences*, *13*(3), 373–396.

[30] D. Otten (2016). A new Lp-antieigenvalue condition for Ornstein–Uhlenbeck operators. *Journal of Mathematical Analysis and Applications*, *444*(1), 136–152. https://doi.org/10.1016/J.JMAA.2016.06.031.

[31] K. Paul (2008). Antieigenvectors of the generalized problem and an operator inequality complementary to Schwarz′s inequality. *Novi Sad Journal of Mathematics*, *38*(2), 25–31.

[32] K. Paul, G. Das & L. Debnath (2015). Computation of antieigenvalues of bounded linear operators via centre of mass. *International Journal of Applied and Computational Mathematics*, *1*(1), 111–119. https://doi.org/10.1007/s40819-014-0007-5.

[33] B. Plestenjak. *MultiParEig*. MATLAB Central File Exchange 2025. https://www.mathworks.com/matlabcentral/fileexchange/47844-multipareig.

[34] C. Radhakrishna Rao (2005). Antieigenvalues and antisingularvalues of a matrix and applications to problems in statistics. *Research Letters in the Information and Mathematical Sciences*, *8*, 53–76. https://doi.org/10.7153/mia-10-45.

[35] E. Ringh & E. Jarlebring (2021). Nonlinearizing two-parameter eigenvalue problems. *SIAM Journal on Matrix Analysis and Applications*, *42*(2), 775–799. https://doi.org/10.1137/19M1274316.

[36] J. I. Rodriguez, J. H. Du, Y. You & L. H. Lim (2021). Fiber product homotopy method for multiparameter eigenvalue problems. *Numerische Mathematik*, *148*, 853–888. https://doi.org/10.1007/s00211-021-01215-6.

[37] K. Ruymbeek, K. Meerbergen & W. Michiels (2022). Subspace method for multiparameter-eigenvalue problems based on tensor-train representations. *Numerical Linear Algebra with Applications*, *29*(5), Article ID: e2439. https://doi.org/10.1002/nla.2439.

[38] M. Seddighin (2004). Antieigenvalue inequalities in operator theory. *International Journal of Mathematics and Mathematical Sciences*, *2004*(57), 3037–3043. https://doi.org/10.1155/S0161171204403615.

[39] M. Seddighin (2005). Computation of antieigenvalues. *International Journal of Mathematics and Mathematical Sciences*, *2005*(5), 815–821. https://doi.org/10.1155/IJMMS.2005.815.

[40] M. Seddighin (2005). On the joint antieigenvalues of operators in normal subalgebras. *Journal of Mathematical Analysis and Applications*, *312*(1), 61–71. https://doi.org/10.1016/j.jmaa.2005.03.021.

[41] M. Seddighin (2009). Antieigenvalue techniques in statistics. *Linear Algebra and Its Applications*, *430*(10), 2566–2580. https://doi.org/10.1016/j.laa.2008.05.007.

[42] M. Seddighin (2012). Approximations of antieigenvalue and antieigenvalue-type quantities. *International Journal of Mathematics and Mathematical Sciences*, *2012*(1), Article ID: 318214. https://doi.org/10.1155/2012/318214.

[43] M. Seddighin & K. Gustafson (2005). On the eigenvalues which express antieigenvalues. *International Journal of Mathematics and Mathematical Sciences*, *2005*(10), 1543–1554. https://doi.org/10.1155/IJMMS.2005.1543.

[44] B. D. Sleeman (1978). Multiparameter spectral theory in Hilbert space. *Journal of Mathematical Analysis and Applications*, *65*(3), 511–530. https://doi.org/10.1016/0022-247X(78)90160-9.

[45] C. Vermeersch & B. De Moor (2022). Two complementary block Macaulay matrix algorithms to solve multiparameter eigenvalue problems. *Linear Algebra and Its Applications*, *654*, 177–209. https://doi.org/10.1016/j.laa.2022.08.014.

[46] H. Volkmer (2006). *Multiparameter Eigenvalue Problems and Expansion Theorems*. Springer, Heidelberg. https://doi.org/10.1007/BFb0089295.